Sharp two parameter bounds for logarithmic and arithmetic-geometric means

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Abstract: For fixed $s \geq 1$ and $t_1, t_2 \in (0, 1/2)$ we prove that the inequalities $G^s(t_1a + (1-t_1)b, t_1b + (1-t_1)a)A^{1-s}(a,b) > AG(a,b)$ and $G^s(t_2a + (1-t_2)b, t_2b + (1-t_2)a)A^{1-s}(a,b) > L(a,b)$ hold for all $a, b > 0$ with $a \neq b$ if and only if $t_1 \geq 1/2 - \sqrt{2s}/(4s)$ and $t_2 \geq 1/2 - \sqrt{6s}/(6s)$. Here $G(a,b)$, $L(a,b)$, $AG(a,b)$ and $A(a,b)$ are the geometric, logarithmic, arithmetic-geometric and arithmetic means of a and b, respectively.
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1 Introduction

For real numbers a, b and c with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \ |x| < 1. \tag{1.1}$$

Here $(a,0) = 1$ for $a \neq 0$, and $(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$ is the shifted factorial function for $n = 1, 2, \cdots$. In connection with the Gaussian hypergeometric function, the well-known complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)(0 < r < 1)$ of the first and second kinds [1, 2] are defined by

$$\begin{cases} \mathcal{K}(r) = \pi F(1/2, 1/2; 1; r^2)/2 = \int_0^{\pi/2}(1 - r^2\sin^2\theta)^{-1/2}d\theta, \\ \mathcal{K}(0) = \pi/2, \qquad \mathcal{K}(1) = \infty \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathcal{E}(r) = \pi F(-1/2, 1/2; 1; r^2)/2 = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta, \\ \mathcal{E}(0) = \pi/2, \qquad \mathcal{E}(1) = 1, \end{cases} \tag{1.3}$$

respectively. The following formulas for $\mathcal{K}(r)$ were presented in [3]:

$$\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - (1 - r^2)\mathcal{K}(r)}{r(1 - r^2)}, \tag{1.4}$$

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1 + r)\mathcal{K}(r). \tag{1.5}$$

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log a - \log b)$ and $A(a,b) = (a+b)/2$ be the classical harmonic, geometric, logarithmic and arithmetic means of two distinct positive real numbers a and b, respectively. Then it is well known that the inequalities $H(a,b) < G(a,b) < L(a,b) < A(a,b)$ hold for all $a, b > 0$ with $a \neq b$.

The classical arithmetic-geometric mean $AG(a,b)$ of two positive number a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$\begin{aligned} &a_0 = a, \qquad b_0 = b, \\ &a_{n+1} = (a_n + b_n)/2 = A(a_n, b_n), \qquad b_{n+1} = \sqrt{a_n b_n} = G(a_n, b_n). \end{aligned}$$

It is well known that inequalities

$$G(a,b) < \sqrt{A(a,b)G(a,b)} < AG(a,b) < A(a,b) \tag{1.6}$$

hold for all $a, b > 0$ with $a \neq b$.

Recently, the harmonic, geometric, logarithmic, arithmetic-geometric and arithmetic means have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [4-13].

The Gaussian identity [3] shows that

$$AG(1, r)\mathcal{K}(\sqrt{1 - r^2}) = \frac{\pi}{2} \tag{1.7}$$

for all $r \in (0, 1)$.

Carlson and Vuorinen [14], and Brackenn [15] proved that

$$L(a,b) < AG(a,b)$$

for all $a, b > 0$ with $a \neq b$. Vamanamurthy and Vuorinen [16] established that

$$AG(a,b) < \frac{\pi}{2} L(a,b)$$

for all $a, b > 0$ with $a \neq b$.

For $t_1, t_2, t_3, t_4 \in (0, 1/2)$, very recently Chu et al. [17, 18] proved that the inequalities

$$G(t_1a + (1-t_1)b, t_1b + (1-t_1)a) > AG(a,b), \tag{1.8}$$

$$H(t_2a + (1-t_2)b, t_2b + (1-t_2)a) > AG(a,b), \tag{1.9}$$

$$G(t_3a + (1-t_3)b, t_3b + (1-t_3)a) > L(a,b), \tag{1.10}$$

and

$$H(t_4a + (1-t_4)b, t_4b + (1-t_4)a) > L(a,b) \tag{1.11}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $t_1 \geq 1/2 - \sqrt{2}/4$, $t_2 \geq 1/4$, $t_3 \geq 1/2 - \sqrt{6}/6$ and $t_4 \geq 1/2 - \sqrt{3}/6$.

Let $t \in (0, 1/2)$, $s \geq 1$ and

$$Q_{t,s}(a,b) = G^s(ta + (1-t)b, tb + (1-t)a)A^{1-s}(a,b). \tag{1.12}$$

Then it is not difficult to verify that

$$Q_{t,1}(a,b) = G(ta + (1-t)b, tb + (1-t)a),$$

$$Q_{t,2}(a,b) = H(ta + (1-t)b, tb + (1-t)a)$$

and $Q_{t,s}(a,b)$ is strictly increasing with respect to $t \in (0, 1/2)$ for fixed $a, b > 0$ with $a \neq b$.

It is natural to ask what are the least values $t_1 = t_1(s)$ and $t_2 = t_2(s)$ in $(0, 1/2)$ such that inequalities $Q_{t_1,s}(a,b) > AG(a,b)$ and $Q_{t_2,s}(a,b) > L(a,b)$ hold for all $a, b > 0$ with $a \neq b$ and $s \geq 1$. The aim of this paper is to answer these questions, our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1. If $t \in (0, 1/2)$ and $s \geq 1$ then the inequality

$$Q_{t,s}(a,b) > AG(a,b) \tag{1.13}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $t \geq 1/2 - \sqrt{2s}/(4s)$.

Theorem 1.2. If $t \in (0, 1/2)$ and $s \geq 1$ then the inequality

$$Q_{t,s}(a, b) > L(a, b) \tag{1.14}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $t \geq 1/2 - \sqrt{6s}/(6s)$.

Remark 1.1. Let $s = 1, 2$ in Theorem 1.1, then inequality (1.13) becomes inequalities (1.8) and (1.9), respectively.

Remark 1.2. Let $s = 1, 2$ in Theorem 1.2, then inequality (1.14) becomes inequalities (1.10) and (1.11), respectively.

2 Lemmas

In order to prove Theorems 1.1 and 1.2 we need two lemmas, which we present in this section.

Lemma 2.1. Let $u \in [0, 1]$, $s \geq 1$ and

$$f_{u,s}(x) = \frac{s}{2}\log(1 - ux^2) - \log\left(\frac{\pi}{2\mathcal{K}(x)}\right). \tag{2.1}$$

Then $f_{u,s} > 0$ for all $x \in (0, 1)$ if and only if $2su \leq 1$.

Proof. From (1.4) and (2.1) one has

$$f'_{u,s}(x) = -\frac{usx}{1 - ux^2} + \frac{\mathcal{E}(x) - (1 - x^2)\mathcal{K}(x)}{x(1 - x^2)\mathcal{K}(x)} = \frac{F_{u,s}(x)}{x(1 - x^2)(1 - ux^2)\mathcal{K}(x)}, \tag{2.2}$$

where

$$F_{u,s}(x) = -sux^2(1 - x^2)\mathcal{K}(x) + (1 - ux^2)[\mathcal{E}(x) - (1 - x^2)\mathcal{K}(x)]. \tag{2.3}$$

It follows from (1.1)-(1.3) and (2.3) together with elaborated computations that

$$\begin{aligned}
&\mathcal{E}(x) - (1 - x^2)\mathcal{K}(x) \\
=&\frac{\pi}{2}\left[\sum_{n=0}^{\infty}\frac{(-1/2, n)(1/2, n)}{(n!)^2}x^{2n} - (1 - x^2)\sum_{n=0}^{\infty}\frac{(1/2, n)^2}{(n!)^2}x^{2n}\right] \\
=&\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(1/2, n)^2}{2n!(n + 1)!}x^{2n+2},
\end{aligned}$$

$$
\begin{aligned}
\frac{2}{\pi}F_{u,s}(x) =& - sux^2(1-x^2)\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n} + (1-ux^2)\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+2} \\
=& - su\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n+2} + su\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n+4} \\
&+ \sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+2} - u\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+4} \\
=& - sux^2 - su\sum_{n=0}^{\infty}\frac{(1/2,n+1)^2}{[(n+1)!]^2}x^{2n+4} + su\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n+4} \\
&+ \frac{x^2}{2} + \sum_{n=0}^{\infty}\frac{(1/2,n+1)^2}{2(n+1)!(n+2)!}x^{2n+4} - u\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+4} \\
=& x^2\left[\frac{1}{2} - su + \sum_{n=0}^{\infty}\frac{(1/2,n)^2A_n}{2(n+1)!(n+2)!}x^{2n+2}\right], \qquad (2.4)
\end{aligned}
$$

where

$$
A_n = su(n+2)(2n+\frac{3}{2}) + (n+\frac{1}{2})^2 - u(n+1)(n+2) > 0. \qquad (2.5)
$$

We divide the proof into two cases.

Case 1.1. $2su \leq 1$. Then (2.2)-(2.5) lead to conclusion that $f_{u,s}(x)$ is strictly increasing in $(0,1)$. Therefore, $f_{u,s}(x) > f_{u,s}(0^+) = 0$ for all $x \in (0,1)$ follows from (1.2) and (2.1) together with the monotonicity of $f_{u,s}(x)$ in $(0,1)$.

Case 1.2. $2su > 1$. Then (2.2)-(2.4) lead to conclusion that there exists $\delta_1 \in (0,1)$ such that $f_{u,s}(x)$ is strictly decreasing in $(0,\delta_1)$. Therefore, $f_{u,s}(x) < f_{u,s}(0^+) = 0$ for all $x \in (0,\delta_1)$ follows from (1.2) and (2.1) together with the monotonicity of $f_{u,s}(x)$ in $(0,\delta_1)$.

Lemma 2.2. Let $u \in [0,1]$, $s \geq 1$, $\mathrm{arctanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ be the inverse hyperbolic tangent function, and

$$
g_{u,s}(x) = \frac{s}{2}\log(1-ux^2) + \log\left(\frac{\mathrm{arctanh}(x)}{x}\right). \qquad (2.6)
$$

Then $g_{u,s}(x) > 0$ for all $x \in (0,1)$ if and only if $3su \leq 2$.

Proof. From (2.6) one has

$$
g'_{u,s}(x) = -\frac{sux}{1-ux^2} + \frac{x-(1-x^2)\mathrm{arctanh}(x)}{x(1-x^2)\mathrm{arctanh}(x)} = \frac{G_{u,s}(x)}{x(1-x^2)(1-ux^2)\mathrm{arctanh}(x)}, \qquad (2.7)
$$

where

$$G_{u,s}(x) = -sux^2(1-x^2)\text{arctanh}(x)+(1-ux^2)[x-(1-x^2)\text{arctanh}(x)]. \quad (2.8)$$

Making use of series expansion and (2.8) we have

$$\begin{aligned}
G_{u,s}(x) =& - sux^2(1-x^2)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}+(1-ux^2)\left[x-(1-x^2)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}\right] \\
=& - su\sum_{n=0}^{\infty}\frac{x^{2n+3}}{2n+1}+su\sum_{n=0}^{\infty}\frac{x^{2n+5}}{2n+1}+(1-ux^2)\sum_{n=0}^{\infty}\frac{2x^{2n+3}}{(2n+1)(2n+3)} \\
=& x^3\left[\frac{2}{3}-su+\sum_{n=0}^{\infty}\frac{B_nx^{2n+2}}{(2n+1)(2n+3)(2n+5)}\right], \qquad (2.9)
\end{aligned}$$

where

$$B_n = 2u(s-1)(2n+5)+2(2n+1) > 0. \qquad (2.10)$$

We divide the proof into two cases.

Case 1.1. $3su \leq 2$. Then (2.7)-(2.10) lead to conclusion that $g_{u,s}(x)$ is strictly increasing in $(0,1)$. Therefore, $g_{u,s}(x) > g_{u,s}(0^+) = 0$ for all $x \in (0,1)$ follows from (2.6) together with the monotonicity of $g_{u,s}(x)$ in $(0,1)$.

Case 1.2. $3su > 2$. Then (2.7)-(2.9) lead to conclusion that there exists $\delta_2 \in (0,1)$ such that $g_{u,s}(x)$ is strictly decreasing in $(0,\delta_2)$. Therefore, $g_{u,s}(x) < g_{u,s}(0^+) = 0$ for all $x \in (0,\delta_2)$ follows from (2.6) and the monotonicity of $g_{u,s}(x)$ in $(0,\delta_2)$.

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Since both $Q_{t,s}(a,b)$ and $AG(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b) \in (0,1)$. Then from (1.5) and (1.7) together with $b/a = (1-x)/(1+x)$ we have

$$\begin{aligned}
\frac{AG(a,b)}{A(a,b)} =& \frac{AG(1,b/a)}{A(1,b/a)} = \frac{\pi}{\mathcal{K}\sqrt{1-(b/a)^2}(1+b/a)} \\
=& \frac{\pi(1+x)}{2\mathcal{K}(2\sqrt{x}/(1+x))} = \frac{\pi}{2\mathcal{K}(x)}. \qquad (3.1)
\end{aligned}$$

It follow from (1.12) and (3.1) that

$$\log\left(\frac{Q_{t,s}(a,b)}{AG(a,b)}\right)=\log\left(\frac{Q_{t,s}(a,b)}{A(a,b)}\right)-\log\left(\frac{AG(a,b)}{A(a,b)}\right)$$
$$=\frac{s}{2}\log\left[1-(1-2t)^2x^2\right]-\log\left[\frac{\pi}{2\mathcal{K}(x)}\right]. \tag{3.2}$$

Therefore, Theorem 1.1 follows from Lemma 2.1 and (3.2).

Proof of Theorem 1.2. Since both $Q_{t,s}(a,b)$ and $L(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a>b$. Let $x=(a-b)/(a+b)\in(0,1)$. Then (1.12) leads to

$$\log\left(\frac{Q_{t,s}(a,b)}{L(a,b)}\right)=\log\left(\frac{Q_{t,s}(a,b)}{A(a,b)}\right)-\log\left(\frac{L(a,b)}{A(a,b)}\right)$$
$$=\frac{s}{2}\log\left[1-(1-2t)^2x^2\right]+\log\left(\frac{\operatorname{arctanh}(x)}{x}\right). \tag{3.3}$$

Therefore, Theorem 1.2 follows from Lemma 2.2 and (3.3).

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